M303 Remote exam practice paper.

Please note that this paper has not gone through our usual checking processes and so may contain mistakes and/or omissions.

Part 1

You should **attempt all questions**.
Each question is worth 11%.

Question 1

(a) Let X_n be the sequence defined by the following recurrence.

$$X_1 = 1, X_2 = 1$$
$$X_{n+1} = X_n + X_{n-1} \qquad \text{for } n \geq 2.$$

Use the Second Principle of Mathematical Induction to prove that $X_{n+5} = X_4 X_n + X_5 X_{n+1}$ [6]

(b) From the definition of congruence alone, prove the following, for integers a, b, c, k and n with $n > 1$ and $k > 0$. If $a \equiv b \pmod{n}$ then $ka + c \equiv kb + c \pmod{n}$. [2]

(c) Suppose that m and n are positive integers such that n has remainder 3 when divided by 4 and that $m = 9n + 4$. Prove that m and n are coprime. [3]

Question 2

(a) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_{11}$.

(i) Write down the order of G. [1]

(ii) Write down the order of $(0, 3)$. [1]

(iii) Write down an element of order 22. [1]

(iv) Is G cyclic? Justify your answer. [1]

(b) Consider the dihedral group D_{22} given by

$$D_{22} = \langle r, s | r^{22} = s^2 = e, sr = r^{21}s \rangle.$$

(i) Write the element $r^{11}sr^{11}$ in the form $r^i s^j$ for some $i \in \{0, \ldots, 21\}$ and some $j \in \{0, 1\}$.

(ii) Find the order of the element $r^{11}s$.

(iii) Is the subgroup $\langle r^{11} \rangle$ normal in D_{22}? Justify your answer. [4]

(c) Let f be an onto homomorphism from a group G to a group H. Prove that if G is abelian then so is H. [3]

Question 3

(a) Given $28 = 2^2 \times 7$, list all the divisors of 28.

Hence, or otherwise, find

(i) $\tau(28)$

(ii) $\sigma(28)$

(iii) $\phi(28)$ [3]

(b) Show that $x^4 + 8$ is irreducible over $\mathbb{Z}$. [4]

(c) Use the rational root test to show that

$$f(x) = x^3 - 4x^2 + 8x + 2$$

is irreducible over $\mathbb{Q}$. [4]

Question 4

Let

$$A = \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4\},$$
$$B = \{(a, b) \in \mathbb{R}^2 : |b| < 1\} \quad \text{and}$$
$$C = A \cap B.$$

(a) Sketch on separate diagrams or describe in words each of A, B and C taking care to indicate which points lie in the sets. [3]

(b) Write down the closure, interior and boundary of C for the Euclidean metric. [3]

(c) Write down the closure and interior of C for the mixed metric, e, given by:

$e\big((x_1, x_2), (y_1, y_2)\big) = d_0(x_1, y_1) + |x_2 - y_2|$ for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. [2]

(d) Determine the e-boundary of $S = \{(0, b) : 0 < b \leq 1\}$ in $\mathbb{R}^2$. [3]

Question 5

Let $\alpha = \sqrt{11} - \sqrt{3}$; you may assume that
$[\mathbb{Q}(\sqrt{11}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$, $\sqrt{11} \notin \mathbb{Q}(\sqrt{3})$ and $\sqrt{3} \notin \mathbb{Q}(\sqrt{11})$.

(a) Express α^3 in the form $a\sqrt{11} + b\sqrt{3}$ for suitable $a, b \in \mathbb{Q}$. [2]

(b) With the help of (a) or otherwise, show that $\sqrt{11}, \sqrt{3} \in \mathbb{Q}(\alpha)$. [3]

(c) Hence show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{11}, \sqrt{3})$. [1]

(d) Determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. [2]

(e) By considering α^2 find the minimal polynomial for α over $\mathbb{Q}$. [3]

Question 6

Define a metric, d, on $X = \mathbb{R}^2 - \{(a, b) : b = 0\}$ by

$$d(\mathbf{x}, \mathbf{y}) = 3|x_1 - y_1| + |x_2 - y_2|$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$.
(You do not have to show that d is a metric.)

(a) Write down $d\big((-1, 1), (1, 1)\big)$. [*1*]

(b) Write down a d-disconnection of X. [*2*]

(c) (i) Show that $\big((0, 1/n)\big)$ is not a d-convergent sequence in X. [*4*]

(ii) Hence, or otherwise, show that X is not d-complete. [*3*]

(iii) Deduce that X is not d-sequentially compact. [*1*]

Part 2

You should **attempt two questions**. If you attempt more, only the scores from your first two questions will count.
Each question is worth 17%.

Question 7

(a) Let G be a group of order $2116 = 2^2 \times 23^2$

(i) Write down the order of the Sylow 23-subgroups and the Sylow 2-subgroups. [*1*]

(ii) Determine the possible numbers of Sylow 2-subgroups and Sylow 23-subgroups. [*4*]

(b) Let G be a non-abelian group of order 2116.

Using part (a), prove that G cannot have a unique Sylow 2-subgroup. [*4*]

(c) Let H be the dicyclic group of order 2116

$$\mathrm{Dic}_{529} = \langle a, b | a^4 = e, a^2 = b^{529}, aba^3b = e \rangle.$$

and let K be the dihedral group of order 2116

$$D_{1058} = \langle r, s | r^{1058} = s^2 = e, sr = r^{1057}s \rangle$$

(i) Write down a group isomorphic to the Sylow 2-subgroup of H.

(ii) Write down a group isomorphic to the Sylow 2-subgroup of K.

(iii) Prove that the centre of H has an element of order 2.

(iv) Determine all elements in the centre of K. [*5*]

(d) Prove that $\mathbb{R}/\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ cannot be isomorphic. [*3*]

Question 8

(a) Let p be an odd prime and let $R = \mathbb{Z}[\sqrt{-2p}]$, where

$$\mathbb{Z}[\sqrt{-2p}] = \left\{ a + b\sqrt{-2p} : a, b \in \mathbb{Z} \right\}.$$

Define a norm, N, on R by $N(a + b\sqrt{-2p}) = a^2 + 2pb^2$. You may assume that N is multiplicative and that any unit has a norm of 1.

(i) Determine all the units of R.

(ii) Determine all elements of R with norm equal to $2p$.

(iii) Prove that the ideal $I = \langle 4, 2 + 2\sqrt{-2p} \rangle$ is not equal to R.

(b) Let α be a root of $x^2 - x - 18$ and let $R = \mathbb{Z}[\alpha]$. You may assume that R is an integral domain, that each element of R can be written as $a + b\alpha$ and that the function $N : R - \{0\} \to \mathbb{N} \cup \{0\}$ given by :

$$N(a + b\alpha) = |a^2 + ab - 18b^2|$$

is a multiplicative and Euclidean norm.

(i) Show that $(943 + 250\alpha)^n$ is a unit in R for any $n \in \mathbb{N}$. [3]

(ii) Explain why every ideal in R is equal to $\langle r \rangle$ for some $r \in R$. [1]

(iii) Let $I = \langle 1 + \alpha \rangle$. Prove that $I = \{a + b\alpha : a \equiv b \pmod{16}, a, b \in \mathbb{Z}\}$. [6]

Question 9

(a) Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + 2|x_2 - y_2|.$$

(i) Prove that d is a metric on $\mathbb{R}^2$ by showing that:

i. (M1) holds [3]

ii. (M2) holds [1]

iii. (M3) holds. [3]

(ii) Find the d-closure of $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ in $\mathbb{R}^2$. [3]

(b) Let e denote the induced Euclidean metric on the interval $[0, 1]$, and let $f\colon [0, 1] \to [0, 1]$ be given by

$$f(x) = \frac{1}{5}x^4 + \frac{3}{4}.$$

(i) Verify that for each $x \in [0, 1]$, $0 \leq f(x) \leq 1$. [1]

(ii) Show that f is a contraction for e.
Hint: you may find it helpful to recall that

$$x^4 - y^4 = (x - y)(x + y)(x^2 + y^2).$$

[2]

(iii) Show that $([0, 1], e)$ is a complete metric space. [2]

(iv) Hence, or otherwise, show that f has a unique fixed point in the interval $[0, 1]$. [2]

[END OF QUESTION PAPER]